ESTIMATING THE INSTABILITIES OF N CORRELATED CLOCKS

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Abstract

The estimation of individual instabilities of N clocks, when only differences of phase-time clocks readings are available, can be carried out even supposing cross-correlation between clocks. Based on the original analysis of this problem, developed by P. Tavella and A. Premoli, and the Kuhn-Tucker theorem (1951), a new way to solve the constrained minimization problem will be introduced. We propose our particular Kuhn-Tucker equations: The already well-known constraint function and two new candidates objective functions. Advantages and inconveniences in the objective function election will be considered. Finally, our proposal will consist of solving two constrained minimization problems: the first of them to obtain an intermediate solution to be used as initial condition in the second minimization problem. Examples with some experimental data from the Real Observatorio de la Armada en San Fernando illustrate the capabilities of this proposal.

INTRODUCTION

It is well known that an useful mathematical tool to characterize the stability of any device is an estimation of the variance by using the available measurements. When the stability of several clocks is evaluated, the absolute variance cannot be directly estimated because the measurement data available are time deviations between pairs of real clocks.

In past years, many methods were introduced to solve this problem: from the classical and popular "three-cornered-hat" method, to the subsequent generalizations to N clocks [1], but all the works assume the hypothesis of independence between clocks (and therefore, the hypothesis of null cross-correlation variables). The problem so resolved occasionally produces negative estimates of variance; this fact may be attributed to several causes, the more notable one to accept incorrelation between clocks.

In a recent work, A. Premoli and P. Tavella [2] expound lifting the assumption of uncorrelation, and proposed a revisited version of the three-cornered-hat method, consistent and formally equivalent to the classical one when clocks are uncorrelated, but that no longer produces negative estimates of variance. Later, the same authors establish the theoretical basis to undertake the problem generalized

Report Documentation Page

Form Approved
OMB No. 0704-0188

1. REPORT DATE	2. REPORT TYPE	3. DATES COVERED
DEC 1999		**00-00-1999 to 00-00-1999**

4. TITLE AND SUBTITLE	5a. CONTRACT NUMBER
Estimating the Instabilities of N Correlated Clocks	5b. GRANT NUMBER
	5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)	5d. PROJECT NUMBER
	5e. TASK NUMBER
	5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) **Real Instituto y Observatorio de la Armada,C/Cecilia Pujazon S/N,,11.100 San Fernando, CBdiz, Spain,**	8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	10. SPONSOR/MONITOR'S ACRONYM(S)
	11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution unlimited

13. SUPPLEMENTARY NOTES
See also ADM001481. 31st Annual Precise Time and Time Interval (PTTI) Planning Meeting, 7-9 December 1999, Dana Point, CA

14. ABSTRACT
see report

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT **unclassified**	b. ABSTRACT **unclassified**	c. THIS PAGE **unclassified**	**Same as Report (SAR)**	**12**	

to N clocks, and demonstrate that the arbitrariness in estimating the solution is reduced when the number of compared clock increases [3].

At 30th PTTI, F. Torcaso, C.R. Ekstrom, E.A. Burt, and D.N. Matsakis proposed a numerical method for the practical resolution of the problem generalized to N clocks [4].

The goal of this work is to propose an alternative method, based on the use of a new objective function, with a better physical meaning. The technique will be applied to simulated and real series of data. The obtained results will be finally discussed.

NOTATION AND DEFINITIONS

The absolute estimated variance can only be calculated from time deviations between a physical clock and an ideal one, but a good approach can be carried out from the existing relations between absolute variances and covariances (jointly called (co)variances) and their corresponding relatives between individual clocks.

Let $\mathbf{X}^i = \{X_k^i : k \geq 1\}$ for $i = 1, \ldots, N$ be the time differences of the i^{th} clock referenced to an ideal one, measured at intervals of τ_0 seconds. Denoting the absolute fractional frequency of the i^{th} clock, averaged over a time $\tau = m\tau_0$ starting at time t_k, by:

$$\bar{Y}_k^i(\tau) = \frac{X_{k+m}^i - X_k^i}{\tau} \tag{1}$$

The Allan variance, or two-sample variance, can be expressed as:

$$r_{ii}(\tau) = \frac{1}{2}\left\langle \left(\bar{Y}_2^i - \bar{Y}_1^i\right)^2 \right\rangle \tag{2}$$

where $\bar{Y}_1, \bar{Y}_2$ denote pairs of adjacent absolute fractional frequencies and $\langle\ \rangle$ denote mathematical expectation or infinite time average. We suppose here, as usual, that the process of averaged fractional frequencies is stationary and ergodic, making the expression (2) well-defined.

Let now $\mathbf{x}^i = \{x_k^i : k \geq 1\}$ for $i = 1, \ldots, N-1$ represent the time differences of the i^{th} clock referred to the N^{th} one taken as a reference; then we can express $\mathbf{x}^i = \mathbf{X}^i - \mathbf{X}^N$. The corresponding fractional frequencies are:

$$\bar{y}_k^i(\tau) = \frac{x_{k+m}^i - x_k^i}{\tau} \tag{3}$$

The absolute and relative fractional frequencies are related by the following expression: $\bar{y}_k^i(\tau) = \bar{Y}_k^i - \bar{Y}_k^N$, $k \geq 1$, and the Allan variance of i^{th} clock referenced to N^{th} clock can be expressed

$$s_{ii}(\tau) = \frac{1}{2}\left\langle \left(\bar{y}_2^i - \bar{y}_1^i\right)^2 \right\rangle \tag{4}$$

by:

Both $s_{ii}(\tau)$ and $r_{ii}(\tau)$ are theoretical quantities impossible to obtain experimentally. They could only be estimated from a finite number of samples $\bar{y}_k^i(\tau)$. The estimate of Allan variance can be obtained by the expression:

$$\hat{s}_{ii}(\tau) = \frac{1}{2(M-2m)m^2\tau_0^2}\sum_{k=1}^{M-2m}\left(x_{k+2m}^i - 2x_{k+m}^i + x_k^i\right)^2 \tag{5}$$

In this paper, the Allan covariance of clocks i^{th} and j^{th} referenced to the N^{th} clock are needed, moreover, we will attempt to estimate the dereferenced Allan covariances between pair of clocks. We define both statistical parameters as they were introduced in [4]:

$$\begin{aligned} s_{ij}(\tau) &= \frac{1}{2}\left\langle \left(\bar{y}_2^i - \bar{y}_1^i\right)\left(\bar{y}_2^j - \bar{y}_1^j\right) \right\rangle \\ r_{ij}(\tau) &= \frac{1}{2}\left\langle \left(\bar{Y}_2^i - \bar{Y}_1^i\right)\left(\bar{Y}_2^j - \bar{Y}_1^j\right) \right\rangle \end{aligned} \tag{6}$$

We introduce again the estimates of both quantities as:

$$\begin{aligned} \hat{s}_{ij}(\tau) &= \frac{1}{2(M-2m)m^2\tau_0^2}\sum_{k=1}^{M-2m}\left(x_{k+2m}^i - 2x_{k+m}^i + x_k^i\right)\left(x_{k+2m}^j - 2x_{k+m}^j + x_k^j\right) \\ \hat{r}_{ij}(\tau) &= \frac{1}{2(M-2m)m^2\tau_0^2}\sum_{k=1}^{M-2m}\left(X_{k+2m}^i - 2X_{k+m}^i + X_k^i\right)\left(X_{k+2m}^j - 2X_{k+m}^j + X_k^j\right) \end{aligned} \tag{7}$$

It is obvious that a covariance is reduced to a variance making $i=j$, so it is easy to derive the expression for $\hat{r}_{ii}$ from $\hat{r}_{ij}$.

In the subsequent paragraphs, we will assume τ as the time interval in which we want to characterize the stability, so we will omit the symbol τ assuming that time interval.

According to the previous expressions, the following relations can be deduced:

$$s_{ij} = r_{ij} + r_{NN} - r_{iN} - r_{jN} \tag{8}$$

and

$$\hat{s}_{ij} = \hat{r}_{ij} + \hat{r}_{NN} - \hat{r}_{iN} - \hat{r}_{jN} \tag{9}$$

These quantities can be jointly denoted as matrices. The N by N matrices $\mathbf{R}$ and $\hat{\mathbf{R}}$ are related to the dereferenced (co)variances; on the other hand, the (N-1) by (N-1) matrices $\mathbf{S}$ and $\hat{\mathbf{S}}$ are related to the (co)variances referenced to a Nth clock.

THE PREVIOUS APROACHES

Before introducing the new proposal of this work, we need to make a review on the previous approaches based on similar arguments, that is: the Tavella-Premoli approach [2], [3] and the F. Torcaso *et al.* approach [4].

In a first approach, Tavella and Premoli propose rejecting the hypothesis of uncorrelated clocks, leading to an underdetermined linear system. They introduce the estimates of covariance s_{ij} and a suitable optimization criterion to calculate the complete covariance matrix $\hat{\mathbf{R}}$. Then, they analyze in detail the case of three clocks, obtaining finally a quasi-analytical solution.

According to their definitions, both $\hat{\mathbf{S}}$ and $\hat{\mathbf{R}}$ are symmetric matrices, so the number of unknown quantities is reduced to $N(N+1)/2$, while the number of equations is $(N-1)N/2$. For the three-clock case, the six unknown quantities are the (co)variances $\hat{r}_{11}, \hat{r}_{22}, \hat{r}_{33}, \hat{r}_{12}, \hat{r}_{13}$, and $\hat{r}_{23}$ and the three equations are:

$$\begin{aligned} s_{11} &= r_{11} + r_{33} - 2r_{13} \\ s_{12} &= r_{12} + r_{33} - r_{13} - r_{23} \\ s_{22} &= r_{22} + r_{33} - 2r_{23} \end{aligned} \tag{10}$$

Nevertheless, they outlined an important constraint which bounds the solution space and guarantees a significant result: the positive definiteness of any covariance matrix. They expressed this constraint as a function $H(r_{13}, r_{23}, r_{33})$, so that the matrix $\hat{\mathbf{R}}$ would be positive-definite, provided that the function $H(r_{13}, r_{23}, r_{33}) > 0$:

$$H(r_{13}, r_{23}, r_{33}) = r_{33} - (r_{13} - r_{33}, r_{23} - r_{33}) S^{-1} (r_{13} - r_{33}, r_{23} - r_{33})^T \tag{11}$$

This function, when equalled to zero, represents the bound of the solution domain, and geometrically it represents an elliptic paraboloid. Afterwards, the authors suggested the determination of the values $\hat{r}_{13}, \hat{r}_{23}$,and $\hat{r}_{33}$ by means of the resolution of a minimization problem, and they defined an objective function $[G(r_{13}, r_{23}, r_{33})]^2$ that is directly proportional to the quadratic sum of covariances and inversely to H:

$$F(r_{13}, r_{23}, r_{33}) = K \frac{[G(r_{13}, r_{23}, r_{33})]^2}{H(r_{13}, r_{23}, r_{33})} \tag{12}$$

Therefore, they minimized the "global correlation" among clocks maintaining the positive definiteness of $\hat{\mathbf{R}}$ leading to the values $\hat{r}_{13}^0$, $\hat{r}_{23}^0$, and $\hat{r}_{33}^0$. The quantities that minimize the objective function are then substituted in (10) concluding the calculation of $\hat{\mathbf{R}}$.

Because of the convex character of the objective function, the solution to the minimization problem is unique. This solution is reduced to the classical three-cornered-hat solution provided that $s_{11} > s_{12}, s_{22} > s_{12}$ and $s_{12} > 0$.

In a second paper, Tavella and Premoli look into the generic problem for N clocks. They demonstrate that the matrix $\hat{\mathbf{R}}$ is positive definite if and only if $|\hat{\mathbf{R}}| > 0$, deriving a compact expression to define the constraint:

$$H(\hat{r}_{1N},\ldots,\hat{r}_{NN}) \equiv \frac{|\hat{\mathbf{R}}|}{|\mathbf{S}|} = \tag{13}$$
$$= \left(\hat{r}_{NN} - \left(\hat{r}_{1N} - \hat{r}_{NN},\ldots,\hat{r}_{(N-1)N} - \hat{r}_{NN}\right)\mathbf{S}^{-1}\left(\hat{r}_{1N} - \hat{r}_{NN},\ldots,\hat{r}_{(N-1)N} - \hat{r}_{NN}\right)^T\right) > 0$$

whose bound can be determined by making $H(\hat{r}_{1N},\ldots,\hat{r}_{NN}) = 0$. This bound represents geometrically an elliptic hyper-paraboloid.

Another interesting conclusion formulated is that the domain delimited by the comparison of N clocks has a higher dimension due to the additional variable $\hat{r}_{(N-1)N}$, but that for any value of this parameter, it can never be larger than the domain obtained from the comparison of N-1 clocks.

The reduction in the amount of arbitrariness in the determination of the covariance matrix $\hat{\mathbf{R}}$ led F. Torcaso *et al.* [4], to think about the generalization of the Tavella-Premoli scheme.

These authors suggest to modify the objective function (12). The new objective function, denoted as the Tavella-Premoli function, is:

$$\frac{\sum_{i<j}\hat{r}_{ij}^2}{H^2(\hat{r}_{1N},\ldots,\hat{r}_{NN})} \tag{14}$$

where H is given by (13). The presence of H squared in the denominator keeps the minimization problem scale-invariant and facilitates the numeric resolution. The initial proposed conditions are selected in order to assure that the initial data lie between the constraint:

$$\hat{r}_{iN}^{ini} = 0 \text{ for } i < N \tag{15}$$
$$\hat{r}_{NN}^{ini} = \frac{1}{2s^*}, \quad s^* = (1,\ldots,1)S^{-1}(1,\ldots,1)^T$$

The minimization problem has a unique solution due to the considerations of convexity already mentioned. The N variables $\hat{r}_{iN} = \hat{r}_{iN}^{0} \quad i = 1,\ldots,N$, can be determined this way. The remaining quantities are determined by resolving the system of $(N-1)N/2$ equations stated in (9).

THE CONSTRAINED MINIMIZATION PROBLEM

The reduction of arbitrariness on the solution domain due to the increasing of the number of clocks is a justified reason in order to look for other valid alternatives that generalize the expounded problem by Tavella and Premoli.

The goal of this new proposal is to reach a solution in close agreement with the physical reality. This can be attained avoiding the distortion of the objective function with the constraint condition. This is the reason we have used the Kuhn-Tucker (*KT*) theorem, published in 1951. This theorem states the following:

"*Given a problem of the type:*

$$\begin{aligned} &min \quad f(\mathbf{x}) \\ &g_i(\mathbf{x}) \le 0, \quad i = 1,\ldots,m \end{aligned} \tag{16}$$

with a local minimum in $\mathbf{x}_{min}$. *If the gradient vectors of the saturated restrictions in* $\mathbf{x}_{min}$ *are linearly independent over* $\mathbf{x}_{min}$, *then each saturated restriction has a positive number* $\lambda_i \ge 0$ *(known as KT multiplier) associated, that verifies:*

$$\nabla f(\mathbf{x}_{min}) + \sum_{i=1}^{m} \lambda_i \cdot \nabla g_i(\mathbf{x}_{min}) = 0 \tag{17}$$

so that, in the previous expression, all the null Multipliers appearing in the restrictions are associated to the not saturated restrictions. In order to get this, the following conditions are imposed:

$$\lambda_i \cdot g_i(\mathbf{x}_{min}) = 0, \quad i = 1,\ldots,m. \tag{18}$$

A restriction is said to be saturated when the optimal solution is verified as equality, otherwise the restriction is said not saturated.

The *KT* equations are necessary conditions for optimality for a constrained minimization problem. If the problem is a so-called convex programming problem, that is, $f(\mathbf{x})$ and $g_i(\mathbf{x}) \quad i = 1,\ldots,m.$ are convex functions, then the *KT* equations are both necessary and sufficient conditions for a global solution point. If the objective function is also strictly convex, in the case the minimum exists that is unique.

The *KT* theorem suggests the employment of a constraint function derived from the (13) condition,

modified appropriately so that make possible the application of the *KT* theorem:

$$G(\hat{r}_{1N},\ldots,\hat{r}_{NN}) \equiv -\frac{H(\hat{r}_{1N},\ldots,\hat{r}_{NN})}{K} < 0 \tag{19}$$

The strict inequality condition that comes in (19) could seem an important limitation, but it is not so from a practical point of view. On the other hand, K denotes $K = \sqrt[N-1]{|\hat{\mathbf{S}}|}$ has been introduced for the sake of adimensionality and to facilitate the numeric resolution. This constraint function represents geometrically the interior of an elliptic hyper-paraboloid, as was seen previously, being therefore a convex function.

Two objective functions could be candidates in the minimization problem, each one with their advantages and disadvantages. The first one seems to be obvious, knowing the antecedents of the problem:

$$F_1(\hat{r}_{1N},\ldots,\hat{r}_{NN}) = \frac{\sum_{i<j}\hat{r}_{ij}^2}{K^2} \tag{20}$$

The second one has greater physical meaning, although it presents certain inconveniences when it is used in a minimization problem:

$$F_2(\hat{r}_{1N},\ldots,\hat{r}_{NN}) = \sum_{i<j}\frac{\hat{r}_{ij}^2}{\hat{r}_{ii}\cdot\hat{r}_{jj}} = \sum_{i<j}\rho_{ij}^2 \tag{21}$$

The convexity of the function done by (20) could be easily demonstrated through the Hessian matrix because the second derivatives are constant. A function that admits derivation partially twice is convex if and only if its Hessian matrix is positive-semidefinite for any value in the domain. If the Hessian matrix is positive-definite, the function will be strictly convex. The obtained Hessian matrix eigenvalues are the following:

$$\begin{aligned} \lambda_i &= 2(N-2), \quad i = 1,\ldots,N-2 \\ \lambda_{N-1} &= \frac{N^2+N-4-\sqrt{N^4+2N^3-15N^2+16N}}{2} \\ \lambda_N &= \frac{N^2+N-4+\sqrt{N^4+2N^3-15N^2+16N}}{2} \end{aligned} \tag{22}$$

For $N \geq 3$, we conclude that F_1 function is strictly convex.

Regarding to the second alternative: F_2, it could not be sure that an unique solution exists. We illustrated an example in Figure 1 in which this fact is stated. In that Figure are shown slices of the elliptic paraboloid, that limits the solution domain for $\hat{\mathbf{S}} = (1.09, 1.18; 1.18, 11.35)$. $\hat{\mathbf{S}}$ has been calculated over three simulated series of gaussian, white frequency noise data, with Allan variances

and correlated coefficients predetermined: $\hat{\sigma}_{11}=2, \hat{\sigma}_{22}=16, \hat{\sigma}_{33}=1$ y $|\rho_{ij}|=0.68\ \forall i\neq j$. Figures 1a, 1b and 1c shown the planes $\hat{\sigma}_{33}=\left(2\cdot(1\,1)\hat{\mathbf{S}}^{-1}(1\,1)^{T}\right)^{-1}=0.54$, $\hat{\sigma}_{33}=0.75$, and $\hat{\sigma}_{33}=1.00$ including the initial point $(0,0,0.54)$ and the prospective solution $(0.97,2.73,1.00)$. It can be seen in each slice the regions where F_2 is positive-definite (+) or it is not (-). It seems to be obvious that the function (21) does not lead to an unique solution. In any case, this function behaves in a soft way, without wide variations.

The last consideration is the main reason why a good election of the initial conditions is considered essential: The closer the initial condition to the final solution is, the higher will be the probability to converge to it.

The final suggestion consists of the following: In a first approximation, equations (20) and (19), with the initial conditions (15), allow calculation of a unique solution (minimum of the quadratic mean of covariances). This solution is used as initial condition over N minimization problems, each one done by a rotation of the N clocks, taking in each case a different clock as reference clock. The value reached by F_2 for the final solution should be lower than the initial value, so the N possible solutions are better than the solution obtained in the first step (from a global minimum of the quadratic mean of correlation coefficients point of view). The last step consist of selecting some of the N solutions; for this, we could apply some validation test able to select the best solution. The used test is based on two criteria:

1. Repetitiousness of each certain final solution
2. Homogeneity of the obtained cross-correlation coefficients.

The first criterion is the more important, its application being enough. When the first criterion fails, the second one, that selects the solution that makes smaller and more homogeneous the absolute values of the correlation coefficients, is applied:

$$\min \quad \frac{\sqrt{\frac{2}{(N-1)N}\sum_{i<j}\left(|\rho_{ij}|-\rho\right)^2}}{\rho}, \quad \text{with } \rho=\frac{2}{(N-1)N}\sum_{i<j}|\rho_{ij}| \tag{23}$$

The robustness of the algorithm could increase considering the N solutions reached when initial conditions (15) are applied minimizing F_2.

If there are any suspicions that two clocks are more or less correlated than the others, F_1 and F_2 might be formulated by introducing a covariance factor for each covariance term.

RESULTS

To illustrate the technique described in the previous section, simulated series of data with a pre-set covariance matrix $\hat{\mathbf{R}}$ were used, having the considered cross-correlation $|\rho_{ij}| = 0.10 \,\forall\, i \neq j$.

We will pay attention over the individual variances specially (diagonal of $\hat{\mathbf{R}}$).

When three clocks were considered, the results obtained were better than those obtained with the two previous approaches, but we have a relative error that comes to be up to 75%. The solution improves drastically when four or more clocks are considered. This can be seen in Table 1, where the results obtained by means of this method and by the previous one proposed by F. Torcaso *et al.*, are indicated, after working the covariance matrix (24) calculated from (7).

$$\hat{\mathbf{S}} = \begin{pmatrix} 2.78 & 0.95 & 2.10 \\ 0.95 & 4.60 & 2.58 \\ 2.10 & 2.58 & 394.57 \end{pmatrix} \tag{24}$$

	$\hat{r}_{11}$	$\hat{r}_{22}$	$\hat{r}_{33}$	$\hat{r}_{44}$
True value	2.03	4.03	397.44	1.02
Estimated by F. Torcaso. *et al.* method.	2.09	3.22	391.43	2.14
New proposal: Intermediate solution.	1.60	2.95	391.77	1.88
New proposal: Final estimate.	1.81	3.64	392.45	0.97

Table 1. Comparison among true and estimated Allan variances.

Table 2 shows the relative error made for each case.

	clock 1	clock 2	clock 3	clock 4
Estimated by F. Torcaso *et al.* method.	0.03	0.20	0.02	1.10
New proposal: Intermediate solution.	0.21	0.27	0.01	0.84
New proposal: Final estimate.	0.11	0.10	0.01	0.05

Table 2. Relative errors ($\frac{|\delta x|}{x_{True}}$).

Tables 1 and 2 don't show significant differences between the estimate proposed by F. Torcaso *et al.* and our intermediate solution, but both estimates introduce high relative errors: the lower the Allan variance is the higher is its relative error. These inadequacies don't seem to exist in our final solution. Nevertheless, we should keep in mind that the simulated data used to calculate the matrix $\hat{\mathbf{S}}$ have equal cross-correlation values (in absolute value), which could explain the good results obtained.

	35 718	14 896	16 113	12 1223	14 1569	31 422	35 583
Referenced to UTC scale	0.14	1.97	29.01	94.61	9.16	5.93	0.47
Estimated by F. Torcaso. *et al.* method.	1.00	2.51	28.59	89.52	9.29	6.35	1.11
New proposal: Intermediate solution.	1.93	1.79	35.75	100.83	6.62	7.37	1.58
New proposal: Final estimate.	0.20	1.58	30.25	93.34	8.65	6.28	0.38

Table 3. Comparison of estimated Allan variances evaluated according to several alternatives.

The technique has also been applied on phase-time deviation data obtained from atomic clocks at the Real Observatorio de la Armada en San Fernando (ROA). Data correspond to seven commercial cesium-beam standards over a period of two years (1996-1997): standards 583 and 718 are HP 5071 high performance models; the 896, 569 and 1569 are HP 5061 Opt.-004; the 1223 was an HP 5061A and the 113 was an OSCILLOQUARTZ model 3200; the series of data were taken at 5-day intervals, with the purpose of knowing the phase-time deviation relative to UTC. The Allan covariance matrix was evaluated for an integration time of 20 days, with overlapping samples; results were compared using the Allan covariance matrix related to the (co)variances referred to the UTC scale (for a time interval τ = 20d., we considered that the UTC scale is quite more stable than any of our clocks, and thus could be supposed quasi-ideal). The selected time interval is considered wide enough so that correlation exists among the clocks. The obtained results are shown in Table 3, where numerical values have been scaled by 10^{28}:

At this point, we would like to emphasize that the time period was selected in this way, because in this epoch old and new clocks cohabited.

When we look at the Table 3 thoroughly, two results stand out from the rest:

1. No prominent difference exists between the procedure described by F. Torcaso *et al.* [4], and the intermediate solution obtained when the constraint condition is separated from the objective function.
2. Results estimated with the new proposal seem to be more in agreement with those "expected": usually, minimizing the quadratic sum of covariances produces a pessimistic estimate of the stability for the most stable clocks when there is strong heterogeneity among them; this doesn't happen with the new proposal.

CONCLUSIONS

Separating the constraint condition of the objective function doesn't seem to introduce important benefits, because the solutions reached in both cases are very similar. Nevertheless, this new estimate truly makes minimum the quadratic sum of covariances.

The resolution of the problem in two phases supposes an important qualitative improvement in the estimation of the solution, since the second objective function used has greater physical meaning. The existence of several minima for this objective function should not be a problem when applying the procedure that has been described above.

The minimization algorithm, as well as the validation tests, is easy to implement, which, along with next to the goodness of the obtained results, could convert it in a very useful method for estimating the frequency stability.

ACKNOWLEDGMENTS

The authors wish to thank to Amadeo Premoli, Politecnico di Milano, and Patrizia Tavella, of IEN, for their valuable comments and advice, and to Demetrio Matsakis, USNO, by providing data of their standards in order to evaluate this new proposal.

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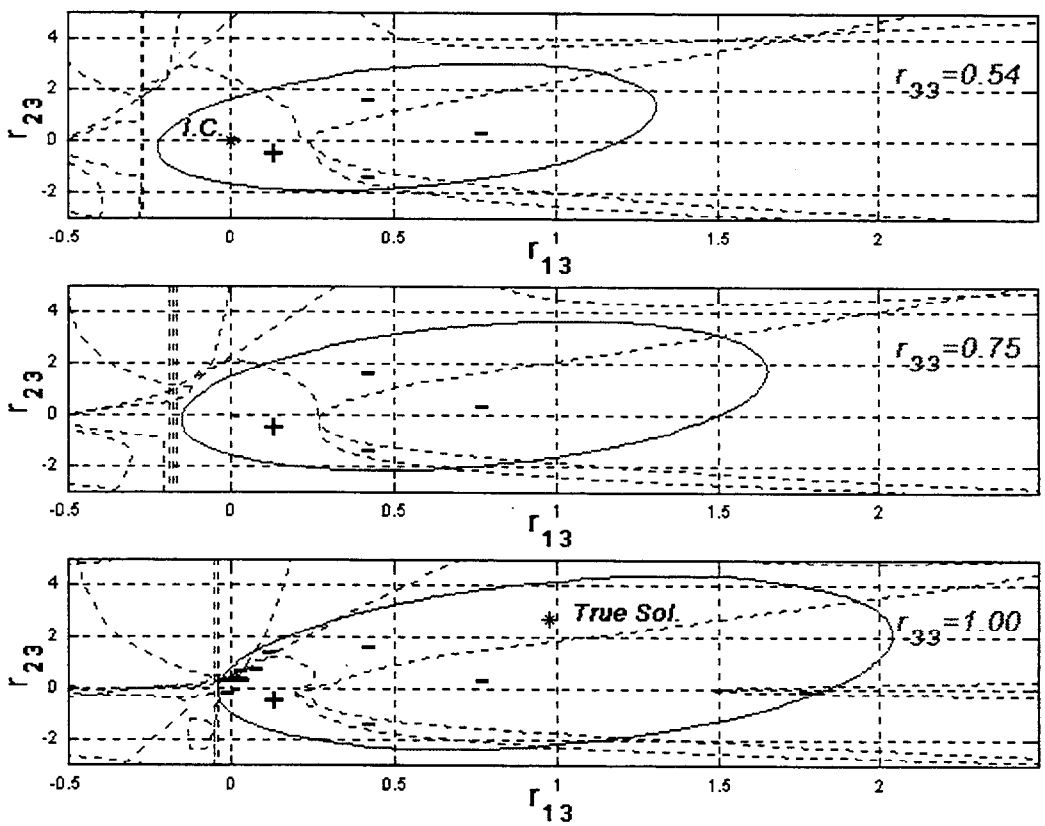

Figure 1. Character of an Hessian matrix corresponding to an particular example for three clocks.